Sr.No. 14336

Exam Code: 211002

Subject Code : 4975

M.Sc (Mathematics) - 2nd Semester

(2721)

Paper: Math-561 Real Analysis-II

Time Allowed: 2 hrs. **Max. Marks: 100**

Note: There are EIGHT questions of equal marks. Candidates are required to attempt any FOUR questions.

Unit-I

1. (a) Prove that the sequence of functions $\{f_n\}$ defined below is pointwise convergent but not uniformly convergent on $(0,1)$.

$$f_n(x) := \begin{cases} 1 & ; x \in (0, 1/n), \\ 0 & ; x \in [1/n, 1). \end{cases}$$

   (b) What is the domain of convergence of the series $\sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n}$?

2. (a) Let X be a countable set and $\{f_n\}$ be a pointwise bounded sequence of real valued functions on X. Then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$, which converges pointwise on X.

   (b) Let Ω be a collection of real valued functions on X. Prove that Ω is equicontinuous on X if and only if $\Omega \setminus F$ is equicontinuous on X for every finite set $F \subset \Omega$.

   (c) State Stone-Weierstrass theorem and provide any one of its applications.

Unit-II

3. (a) Let E denote the set of points in $(0,1)$ whose decimal representation doesn't contain digits 0 and 9. Is E measurable? Obtain $m^*(E)$.

   (b) What are Borel sets? Are these countably many? Does there exist a Borel set which is not measurable? Justify your answers.

4. (a) For $A \subset \mathbb{R}$ and $x \in \mathbb{R}$, define $xA := \{xa : a \in A\}$. Prove that $m^*(xA) = xm^*(A)$. If A is measurable, prove that xA is also measurable.

   (b) Prove that $E \subset \mathbb{R}$ is Lebesgue measurable if and only if there exists a G_δ-set $G \supset E$ such that $m^*(G \setminus E) = 0$.

Unit-III

5. (a) Let $f : \mathbb{R} \longrightarrow [0, +\infty)$ be a measurable function. Prove that $f = \sum_{n=1}^{\infty} s_n$, for some sequence $\{s_n\}$ of simple functions.

   (b) Let $\{f_n\}$ be a monotone sequence of non-negative measurable functions, which converges pointwise a.e. to a function f. Under which hypothesis, can one conclude that $\int f = \lim_{n\to\infty} \int f_n$? Justify.

Contd....P/2

6. (a) Prove that every (bounded) Riemann integrable function is Lebesgue integrable.

   (b) If f is a Lebesgue integrable real function, prove that so is $|f|$ and $|\int f| \leq \int |f|$. Show that the converse is false, in general. Under which hypothesis the converse holds true? Justify your answer.

Unit-IV

7. (a) Discuss the inclusion relation between the classes of absolutely continuous, Lipschitz continuous and uniformly continuous functions. Also provide specific examples of functions which conclude that none of these classes are same.

   (b) Give an example of a continuous function on $[-1, 1]$, which is not of bounded variation.

8. (a) Let $F : [a, b] \longrightarrow \mathbb{R}$ be a measurable function such that F is differentiable a.e. on $[a, b]$. Prove that F' is a measurable function.

   (b) Let $F : [a, b] \longrightarrow \mathbb{R}$ be differentiable on $[a, b]$ and F' is bounded. Prove that F' is Lebesgue integrable and $\int_a^b F' = F(b) - F(a)$.

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